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SOME PROPERTIES OF THE RANK AND INVARIANT FACTORS OF MATRICES*

by

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1. Introduction

Some years ago, G. Pall observed that the invariant factors of the incidence matrices of a certain pair of non-isomorphic projective planes of order 9 were different. With the aim of investigating such phenomena experimentally, we have constructed a code to calculate the invariant factors of rational integral matrices (actually, we compute the Smith's normal form of these matrices, as described, e.g., in MacDuffee [2; p. 41]), and this note is in the nature of a report on some preliminary experiments in the use of this code. In particular, we have computed the invariant factors of all (0, 1) matrices of order ≤ 8, with constant row and column sums, and these data are presented in the Appendix.

An examination of these data suggested three conjectures, all of

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which turned out to be true, and one of which suggested some interesting questions concerning the imbedding of a non-singular matrix in a doubly stochastic matrix of the same rank. The proofs of the conjectures (Remark 1, Remark 2 and Theorem 2) and the discussion of the questions of imbedding (Theorem 1, Theorem 3 and Theorem 4) form the main part of the note. We hope that others may discern additional facts from the data tabulated in the Appendix.

2. Some Simple Remarks

Let $\mathfrak{A}$ be the class consisting of all $m \times n$ $(0, 1)$ matrices with prescribed row sums and column sums.

Remark 1. The ranks of matrices in $\mathfrak{A}$ assume all integers between the minimum rank and the maximum rank of matrices in $\mathfrak{A}$.

Proof: Let A be a matrix in $\mathfrak{A}$. Consider the 2×2 submatrices of A of the types

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

An <u>interchange</u> is a transformation of the elements of A that changes a minor of type A_1 into type A_2 or vice versa and leaves all other elements of A unaltered. The interchange theorem of H. J. Ryser [3] states that if A and B belong to $\mathfrak{A}$, then A is transformable into B by a finite sequence of interchanges. We note that if B is a matrix obtained from A by an interchange, then $B = A + C$ where C

is a matrix whose entries are all zero except for a minor of the form $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, and therefore rank $B \leq$ rank $A + 1$. Similarly, rank $A \leq$ rank $B + 1$. Hence, the rank is altered by at most 1 by an interchange, and the statement to be proven follows at once by the interchange theorem of Ryser.

From now on, let J be a matrix of the appropriate size whose entries are all 1, and let $\widetilde{A} = J - A$.

Remark 2. Let A be any $n \times n$ matrix with rational integral entries. Then the number of units among the invariant factors of A and the number of units among the invariant factors of $\widetilde{A}$ differs by at most 1.

Proof: Let A_1 be the matrix obtainable from A by subtracting the first column from every other column, and let $\widetilde{A}_1$ be the matrix obtainable from $\widetilde{A}$ correspondingly. Except possibly for the first columns, A_1 and $\widetilde{A}_1$ are the same except for sign, and therefore if $e_1, e_2, \ldots, e_n$, with $e_1 | e_2 | \ldots | e_n$, are the invariant factors of A_1 and $\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_n$, with $\widetilde{e}_1 | \widetilde{e}_2 | \ldots | \widetilde{e}_n$, are the invariant factors of $\widetilde{A}_1$, then

$$\prod_{m=1}^{k} e_m \,\Big|\, \prod_{m=1}^{k+1} \widetilde{e}_m \quad \text{and} \quad \prod_{m=1}^{k} \widetilde{e}_m \,\Big|\, \prod_{m=1}^{k+1} e_m, \qquad k = 1, 2, \ldots, n-1.$$

Suppose e_i is the first non-unit invariant factor of A_1 and $\widetilde{e}_j$ is the first non-unit invariant factor of $\widetilde{A}_1$. We may as well assume that $i \leq j$ and $j > 2$. Taking $k = j - 2$, we have

$$e_{j-2}\left|\prod_{m=1}^{j-2} e_m\right| \left|\prod_{m=1}^{j-1} \tilde{e}_m\right| = \pm 1$$

and therefore $j - 1 \leq i \leq j$, which completes the proof.

3. Imbedding Questions

Let $\mathfrak{A}(n,k,p)$ denote the class of all $n \times n$ matrices with real entries whose row and column sums are all $k \neq 0$, and whose rank is $p \leq n$. We shall write $B < \mathfrak{A}(n,k,p)$ for a non-singular matrix B of order p if there exists a matrix $A \in \mathfrak{A}(n,k,p)$ which contains B as a submatrix. We derive as Theorem 1 a necessary condition for $B < \mathfrak{A}(n,k,p)$. As a corollary, we prove a relation between the invariant factors of a rational integral square matrix A and the invariant factors of $J - A$. In Theorem 4, we show that this necessary condition is also sufficient to imbed a rational integral matrix B in a rational integral matrix $A \in \mathfrak{A}(n,k,p)$. We also show in Theorem 3 a set of necessary and sufficient conditions for imbedding a non-negative matrix B in a non-negative matrix $A \in \mathfrak{A}(n,1,p)$, i.e., a doubly stochastic matrix of rank p.

Theorem 1. If $B < \mathfrak{A}(n,k,p)$, then the sum of the elements of B^{-1} is $\frac{n}{k}$.

Proof: Assume

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix},$$

where $A \in \mathfrak{A}(n,k,p)$. Since A is of rank p,

$$E = D B^{-1} C. \tag{3.1}$$

Let u_t be the column vector with t coordinates, each of which is unity. Since the row sums of A are k, we have

$$B u_p + C u_{n-p} = k u_p, \tag{3.2}$$

$$D u_p + E u_{n-p} = k u_{n-p}. \tag{3.3}$$

Inserting (3. 1) and (3. 2) in (3. 3), we obtain

$$D B^{-1} u_p = u_{n-p}. \tag{3.4}$$

Since the column sums of A are k, we have

$$u_p' B + u_{n-p}' D = k u_p'. \tag{3.5}$$

Multiplying both sides of (3. 5) on the right by $B^{-1} u_p$, and substituting in (3. 4), we obtain

$$p + n - p = k u_p' B^{-1} u_p,$$

which was to be proved.

Theorem 2. Let A be a matrix of order n, with row and column sums k, $0 \neq k \neq n$, whose entries are rational integers, and let $\widetilde{A} = J - A$. Let $\widetilde{e}_1, \ldots, \widetilde{e}_n$ be the invariant factors of $\widetilde{A}$; $e_1, \ldots, e_n$ the invariant factors of A. Then

$$\frac{\prod_{e_i \neq 0} e_i}{\prod_{\widetilde{e}_i \neq 0} \widetilde{e}_i} = \frac{k}{n-k} \tag{3.6}$$

up to a unit ± 1.

Proof: Let A be of rank p, B a non-singular matrix of order p contained in A, and $\widetilde{B} = J - B$. We first show that the rank of $\widetilde{B}$ is equal to the rank of $\widetilde{A}$ and that $|\widetilde{B}| \neq 0$. Write

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}, \quad \widetilde{A} = J - A = \begin{pmatrix} \widetilde{B} & \widetilde{C} \\ \widetilde{D} & \widetilde{E} \end{pmatrix}.$$

Since the column sums of A are all k, we have from (3.1)

(3.7) $$u_p' C + u_{n-p}' D B^{-1} C = k u_{n-p}'.$$

Substituting from (3.5), we have

(3.8) $$u_p' B^{-1} C = u_{n-p}'.$$

Let X be a matrix with p rows and $n-p$ columns such that

(3.9) $$BX = C.$$

From (3.8), we have

(3.10) $$u_p' X = u_{n-p}'.$$

It is then clear from (3.9) and (3.10) that

(3.11) $$\widetilde{B} X = \widetilde{C}.$$

Further, since $DX = E$, from (3.1), it follows from (3.10) that

(3.12) $$\widetilde{D} X = \widetilde{E}.$$

In other words, the last $n-p$ columns of $\widetilde{A}$ are linear combinations of the first p columns. Similarly, we can see that the last $n-p$ rows of $\widetilde{A}$ are linear combinations of the first p rows. Consequently, rank

$\widetilde{A}$ = rank $\widetilde{B} \leq p$ = rank B = rank A. But, symmetrically, rank $A \leq$ rank $\widetilde{A}$, which implies rank A = rank $\widetilde{A}$ and therefore rank $\widetilde{B} = p$, from which $|\widetilde{B}| \neq 0$ follows. Now, we have from Theorem 1 that

$$u_p{}'B^{-1}u_p = \frac{n}{k}, \quad u_p{}'\widetilde{B}^{-1}u_p = \frac{n}{n-k}. \tag{3.13}$$

We shall use (3.13) to prove that

$$\frac{|B|}{|\widetilde{B}|} = (-1)^{p-1}\frac{k}{n-k}. \tag{3.14}$$

Clearly, (3.14) implies (3.6), for the numerator of the left-hand side of (3.6) is the g.c.d. of the determinants of order p contained in A, and the denominator is the g.c.d. of the corresponding determinants in $\widetilde{A}$. To prove (3.14), observe first that

$$u_p{}'B^{-1}u_p = u_p{}'(B^{-1}u_p) = \frac{1}{|B|}\sum_j \Delta_j, \tag{3.15}$$

where Δ_j is the determinant of the matrix whose kth column, $k \neq j$, is B_k, the kth columns of B, and whose jth column is u_p. Now,

$$\sum_j \Delta_j = |u_p, B_2 - B_1, \ldots, B_p - B_1|, \tag{3.16}$$

which can be verified from the expansion of the right-hand side. Further, if we apply the same observations to $\widetilde{B}$, we have

$$u_p{}'\widetilde{B}^{-1}u_p = \frac{1}{|\widetilde{B}|}\sum_j \widetilde{\Delta}_j, \tag{3.17}$$

where $\widetilde{\Delta}_j$ is defined in the obvious way, and

$$\sum_j \tilde{\Delta}_j = |u_p, \tilde{B}_2 - \tilde{B}_1, \ldots, \tilde{B}_p - \tilde{B}_1|. \tag{3.18}$$

But $\tilde{B}_k - \tilde{B}_1 = -(B_k - B_1)$, and, therefore, from (3.18) and (3.16), we have

$$\sum_j \tilde{\Delta}_j = (-1)^{p-1} \sum_j \Delta_j. \tag{3.19}$$

Finally, (3.14) follows at once from (3.19), (3.17), (3.15) and (3.13).

Theorem 3. Let B be a non-singular matrix of order p whose entries are non-negative real numbers. In order that B be a submatrix of a doubly stochastic matrix A of order n and rank p, where $p < n \leq 2p - 1$, it is necessary and sufficient that

$$u_p' B^{-1} u_p = n, \tag{3.20}$$

$$\sum_j b_{ij} \leq 1, \quad i = 1, \ldots, p; \quad \sum_i b_{ij} \leq 1, \quad j = 1, \ldots, p, \tag{3.21}$$

$$\sum_{i,j} b_{ij} \geq 2p - n. \tag{3.22}$$

Proof. The necessity of (3.20) is contained in Theorem 1. The necessity and sufficiency of (3.21) and (3.22) in order to effect the imbedding in a doubly stochastic matrix without considering the rank was pointed out in Dulmage and Mendelsohn [1]. To prove the sufficiency of (3.20) - (3.22) with the rank taken into consideration, write

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix},$$

where C, D and E are to be determined. Let each column of C be $\frac{1}{n-p}(I-B)u_p$, each row of D be $\frac{1}{n-p} u_p'(I-B)$, and each entry of E be

$\frac{1}{(n-p)^2} \left(\sum_{i,j} b_{ij} + n - 2p \right)$. Then the non-negativity of the entries of A follows from (3.21) and (3.22) and that A is doubly stochastic is easy to verify. To show that the rank of A is equal to the rank of B, we must see that

$$\frac{1}{(n-p)^2} (u_p' B u_p + n - 2p) = \frac{1}{n-p} u_p' (I-B) B^{-1} \frac{1}{n-p} (I-B) u_p,$$

which follows from (3.20).

Theorem 4. If B is a non-singular matrix of order p with rational integral entries, and the sum of the coefficients of B^{-1} is $\frac{n}{k}$, $n > p$, then B is contained in a matrix of order n and rank p, with rational integral entries, whose row and column sums are all k.

Proof: We wish to find a matrix

$$A = \begin{pmatrix} B & C \\ & F \end{pmatrix}$$

with the desired properties. Let all columns of C other than the last be the same as columns of B, and choose the last column in such a way that the row sums, for each of the first p rows, shall be k. Next, let all rows of F other than the last be the same as rows of (B, C). Choose the last row of F in such a way that all column sums of A are k. All that needs to be checked is that the rank of A is p, which can be done as in the previous theorem.

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APPENDIX

How to read the table:

$$(n, k) \rightarrow (n,\ n-k)$$

$$g_1, g_2, \ldots, g_n \qquad h_1, h_2, \ldots, h_n$$

means that if A, a nxn matrix of zeros and ones whose row sums and column sums are all k, has invariant factors $g_1, g_2, \ldots, g_n$, then J-A has invariant factors $h_1, h_2, \ldots, h_n$.

$$(n, k) \rightarrow (n,\ n-k)$$

$$g_1 g_2, \ldots, g_n \qquad \begin{cases} h_1, h_2, \ldots, h_n \\ h'_1, h'_2, \ldots, h'_n \end{cases}$$

means that if A is the same matrix as described above, then J-A has invariant factors either $h_1, h_2, \ldots, h_n$ or $h'_1, h'_2, \ldots, h'_n$.

(4. 1)				$\rightleftarrows$	(4. 3)			
1	1	1	1		1	1	1	3
(4. 2)				$\rightleftarrows$	(4. 2)			
1	1	0	0		1	1	0	0
1	1	1	0		1	1	1	0

(5. 1)					$\rightleftarrows$	(5. 4)				
1	1	1	1	1		1	1	1	1	4
(5. 2)					$\rightleftarrows$	(5. 3)				
1	1	1	2	0		1	1	1	3	0
1	1	1	1	2		1	1	1	1	3

(6.1)						⇄	(6.5)					
1	1	1	1	1	1		1	1	1	1	1	5

(6.2)						⇄	(6.4)					
1	1	1	0	0	0		1	1	2	0	0	0
1	1	1	1	0	0		1	1	1	2	0	0
1	1	1	1	1	0		1	1	1	1	2	0
1	1	1	1	2	2		1	1	1	1	1	8

(6.3)						⇄	(6.3)					
1	1	0	0	0	0		1	1	0	0	0	0
1	1	1	0	0	0		1	1	1	0	0	0
1	1	1	1	0	0		1	1	1	1	0	0
1	1	1	1	1	0		1	1	1	1	1	0
1	1	1	1	1	9		1	1	1	1	1	9

(7.1)							⇄	(7.6)						
1	1	1	1	1	1	1		1	1	1	1	1	1	6

(7.2)							⇄	(7.5)						
1	1	1	1	2	0	0		1	1	1	1	5	0	0
1	1	1	1	1	2	0		1	1	1	1	1	5	0
1	1	1	1	1	1	2		1	1	1	1	1	1	5

(7.3)							⇄	(7.4)						
1	1	1	1	3	0	0		1	1	1	1	4	0	0
1	1	1	1	1	3	0		1	1	1	1	1	4	0
1	1	1	1	1	1	3		1	1	1	1	1	1	4
1	1	1	1	2	2	6		1	1	1	2	2	2	4

(8.1) $\rightleftarrows$ (8.7)

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	7

(8.2) $\rightleftarrows$ (8.6)

1	1	1	1	0	0	0	0	1	1	1	3	0	0	0	0
1	1	1	1	1	0	0	0	1	1	1	1	3	0	0	0
1	1	1	1	1	1	0	0	1	1	1	1	1	3	0	0
1	1	1	1	1	1	1	0	1	1	1	1	1	1	3	0
1	1	1	1	1	2	2	0	1	1	1	1	1	1	12	0
1	1	1	1	1	1	2	2	1	1	1	1	1	1	1	12

(8.3) $\rightleftarrows$ (8.5)

1	1	1	1	3	0	0	0	1	1	1	1	5	0	0	0
1	1	1	1	1	3	0	0	1	1	1	1	1	5	0	0
1	1	1	1	1	1	3	0	1	1	1	1	1	1	5	0
1	1	1	1	1	1	6	0	1	1	1	1	1	1	10	0
1	1	1	1	1	1	1	3	1	1	1	1	1	1	1	5
1	1	1	1	1	1	1	6	1	1	1	1	1	1	1	10
1	1	1	1	1	1	1	15	1	1	1	1	1	1	5	5
1	1	1	1	1	1	3	3	1	1	1	1	1	1	1	15
1	1	1	1	1	3	3	3	1	1	1	1	1	1	3	15

(8.4)								→ (8.4)							
1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0	1	1	1	0	0	0	0	0
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1	1	1	1	1	2	0	0	1	1	1	1	1	2	0	0
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								1	1	1	1	2	2	0	0
1	1	1	1	2	2	0	0	1	1	1	1	1	4	0	0
1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	0
1	1	1	1	1	1	2	0	1	1	1	1	1	1	2	0
1	1	1	1	1	1	4	0	1	1	1	1	1	1	4	0
								1	1	1	1	1	2	2	0
1	1	1	1	1	1	8	0	1	1	1	1	1	1	8	0
								1	1	1	1	1	2	4	0
1	1	1	1	1	1	16	0	1	1	1	1	1	4	4	0
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1	1	1	1	1	2	4	0	1	1	1	1	1	1	8	0
1	1	1	1	1	4	4	0	1	1	1	1	1	1	16	0
1	1	1	1	1	1	1	16	1	1	1	1	1	1	2	8
								1	1	1	1	1	1	4	4
1	1	1	1	1	1	1	32	1	1	1	1	1	1	1	32
1	1	1	1	1	1	2	8	1	1	1	1	1	1	1	16
1	1	1	1	1	1	4	4	1	1	1	1	1	1	1	16

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